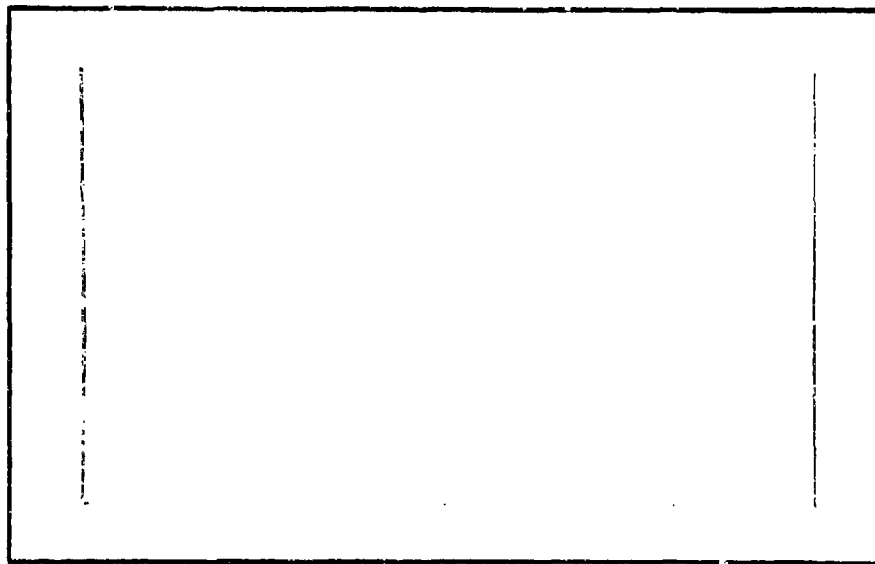AD743265
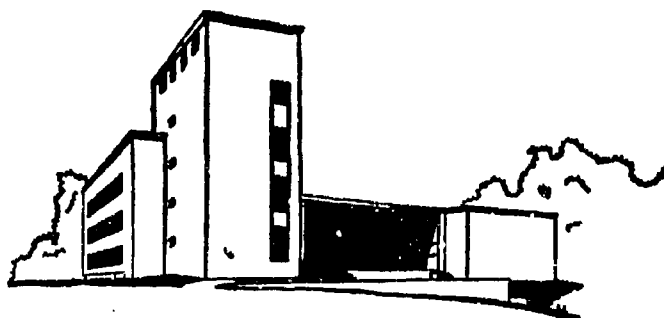
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The name polar programming applies to a general class of non-linear mathematical programming problems which can be solved by the polaroid approach. In integer programming polaroids yield non-trivial extensions of the intersection cut approach.

This paper builds on the properties of polaroid sets (particularly complete convex polaroids) and focuses on the following intersection problem:

Given a point $\bar{x}$ belonging to the polaroid set P*, find the intersection point u* of a one-dimensional ray u with the boundary of P* $u* \in (\text{bd } P* \cap u)$.

We also present a theoretical comparison of the relative merits of several currently proposed cutting planes for integer and/or concave programming.

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On Polaroid Intersections

by Claude-Alain Burdet

0) Introduction

Polaroid sets and functions have been introduced in [9], where we mentioned areas of application in non-linear programming, particularly in quasi-concave and integer optimization problems over a linearly constrained set of feasible solutions.

Polaroids can be used to improve the efficiency of the algorithms outlined by Hoang Tuy in [16] for the maximization (minimization) of a quasi-convex (concave) objective function over a polyhedral set. They also improve in a similar way the modified version of this approach, presented in [12] by Glover and Klingman. Further algorithms outlined in [8] can also be extended by using polaroids and in [10] we gave the name polar programming to a general class of non-linear mathematical programming problems which can be solved by the polaroid approach.

In integer programming polaroids yield non-trivial extensions of the intersection cut approach of Balas [1, 11, 5] as indicated in [9]; in [7], a special type of polaroid is presented and we show how it brings under one roof the enumerative approach of [5,6] and the extension method of Balas [2,3]; it also illustrates a connection between concave and integer programming, of a nature different from that of Rajavaghari [14].

Consider the following general principle for solving the (arbitrary) mathematical programming problem:

$$\text{maximize} \quad g(x), \text{ subject to } x \in P$$

where $\quad x$ is an n vector

$\qquad P$ a subset of $\mathbb{R}^n$

$\qquad g$ a real valued function of $x \in \mathbb{R}^n$.

Method: Construct a (finite) collection of n-dimensional simplices $S_j \subset \mathbb{R}^n$, $j \in J$ such that

a) $P \subset S = \bigcup_{j \in J} S_j$

b) $\forall j \in J$ one has

$$\max_{x \in (S_j \cap P)} g(x) \leq \Delta_j$$

where Δ_j is a _lower_ bound:

$$\Delta_j \leq g(x) \quad , \text{ for some } x \in P$$

One easily establishes that the optimal solution $\tilde{x}$ of the original problem is delivered by (finite) search: $\tilde{x} \in \{x^j, j \in J\}$ such that $g(\tilde{x}) \geq g(x^j)$, $\forall j \in J$.

Particular examples of this general method are described in [8] and [10]; they can be classified as follows:

- _cutting plane_ algorithms (see, for instance, [16], [1], [11], [5], [6], [7])

- _enveloping_ algorithms (see [10])

- _partitioning_ algorithms (see [10], [16], [4], [12])

One of the basic ingredients in the above general method lies in the construction of the simplex S_j , $\forall j \in J$ i.e., in the determination of its (n+1) vertices.

This paper builds on the properties of polaroid sets (particularly com-plete convex polaroids, as defined in [9]) which allow for the construction of simplices S_j possessing the desired properties. We focus our atten-tion on the determination of the vertices of S_j, obtained here from the following intersection problem:

Given a point $\bar{x}$ belonging to the polaroid set P^* (for a definition see section 1 below), find the intersection point u^* of a one-dimensional ray u, with the boundary of P^* : $u^* \in (bd\ P^* \cap u)$.

First we outline the basic methodology which leads to polaroid inter-section algorithms. In a second part we sketch some typical examples of polaroid functions and sets.

The interested reader is referred to [3,4,7,12,19] for application areas and to [9] and [10] for further aspects of the theory of polaroids.

1) Definitions

In order to make this report self contained, we briefly reproduce here some definitions from [9].

Let the polaroid function $f = f(x,y)$ be real valued with two arguments x and y which are n-vectors; let P denote a closed set in $\mathbb{R}^n$.

Definition 1: The polaroid set $P^*(k)$ defined by the polaroid function f with respect to the set P and the parameter $k \in f(P, R^n)$ is

$$P^*(k) = \{y \mid f(x,y) \le k \quad , \quad \forall x \in P\}$$

One has the

Theorem 1: The <u>polaroid set</u> $P^*(k)$ is <u>convex</u> $\forall k$

iff the <u>polaroid function</u> $f = f(x,y)$ is

<u>quasi-convex</u> in y, for all $x \epsilon P$.

Proof: Presented in [9] and omitted here.

Although non-convex polaroid sets P^* may also be of occasional interest (see [9]), we are mainly concerned here with convex polaroids.

Definition 2: Consider the point $\bar{x} \epsilon P^*(k)$ and a vector $a \epsilon \mathbf{R}^n$; we

define the ray u, as the following half-line originating

at $\bar{x}$ in the direction $(-a)$:

$$u = \{u(\lambda) \mid u = \bar{x} - \lambda a \quad, \quad \lambda \geq 0\}$$

Denote by $F_a = F_a(\lambda)$ the real valued function

$$F_a(\lambda) = \max_{x \epsilon P} \{f(x, \bar{x} - \lambda a)\}$$

Proposition 1: Let λ^* satisfy

1) $\lambda^* \geq 0$

2) $F_a(\lambda^*) = k$

then $u^* = u(\lambda^*)$ lies on the "boundary" of $P^*(k)$, i.e.

$$u^* \epsilon \text{ bd } P^*(k) = \{y \epsilon P^*(k) \mid f(x,y) = k \text{, for some } x \epsilon P\}$$

Proof: From the hypothesis 2), $\exists x^*$ such that

$$f(x^*, \bar{x} - \lambda^* a) = f(x^*, u^*) = k \ . \hspace{3cm} \text{QED}$$

We now briefly indicate below (a more detailed study can be found in [9]) how polaroids find their justification and application in non-linear optimization problems: the following results make it possible to solve (in principle) a very broad class of problems using polaroids and the general method outlined above.

Definition 3: The polaroid $P^*(k)$ is called <u>complete</u> if $P \subset P^*(k)$.

Theorem 2: Let $\Delta = g(\widetilde{x}) = \max\limits_{x \in P} g(x)$, with $g(x) = f(x,x)$. If $P^*(\Delta)$ is complete

then every optimal solution $\widetilde{x}$ of the problem

maximize $g(x)$, subject to $x \in P$

lies on the 'boundary' of $P^*(\Delta)$.

Proof: Presented in [9].

Proposition 2: Assume $f(x,y)$ quasi-convex in y , $\forall x \in P$; then $P^*(\Delta)$ is

a <u>complete convex polaroid</u> and any collection of simplices

S_j , $j \in J$ such that $S_j \subset P^*(\Delta)$ $\forall j \in J$ and $P \subset S = \bigcup\limits_{j \in J} S_j$

yields <u>a sufficient optimality condition</u>.

Proof: Convexity of $P^*(\Delta)$ was established in Theorem 1 and completeness

follows from the choice of Δ ; optimality of $\widetilde{x}$ then follows from

the observation that $f(x,y) \leq \Delta = g(\widetilde{x})$, $\forall x \in P$, $\forall y \in S_j$, $\forall j \in J$

and from the assumption $P \subset S$. QED

The general method described in the introduction now becomes operational

when the simplices S_j are constructed as subsets of the polaroids $P^*(k)$,

because, for $k \leq 0$, one has $S_j \subset P^*(k) \subset P^*(\Delta)$ (a proof is given in [9]);

in practice, the value of k is gradually increased (step-wise), until the

optimal value Δ is reached. Thus the only problem remaining in the

design of an algorithm is the practical construction of the simplices S_j .

2) The Intersection Method

The proposition 1 indicates that the determination of intersection

point reduces to the <u>parametric linearly constrained optimization</u>

process described below:

Find $\bar{\lambda} > 0$ such that $F_a(\bar{\lambda}) = \max\limits_{x \in P} f(x, \bar{x} - \bar{\lambda}a) \leq k$;

if $F_a(\bar{\lambda}) < k$ then $\bar{u} = u(\bar{\lambda})$ can be seen to lie in the "interior" of P^*:

Int $P^*(k) = \{y \mid f(x,y) < k , \forall x \in P\}$;

if $F_a(\bar{\lambda}) = k$, then u lies on the "boundary" set bd $P^*(k)$ defined in section 1 and it is therefore a solution u^* to the intersection problem.

The polaroid intersection problem therefore can be seen to contain the following two parts:

P1) A Mathematical programming part:

maximize $z(x) = f(x, \bar{x} - \bar{\lambda}a)$, $\bar{\lambda}$ given

subject to $x \epsilon P$

P2) A parametric search problem:

Find $\lambda^* = \max_{\lambda > 0} \{\lambda \mid F_a(\lambda) \leq k\}$

In general neither of these optimizations can be executed exactly and one often prefers a good approximate solution which is obtained in few computations. This approximation, however, must be such that the approximate intersection point $\tilde{u}$ belongs to the set $P^*(k)$.

For the problem P1) this means that one needs an upper bound solution $\tilde{x}$ such that

$$z(x) \leq \tilde{z} = z(\tilde{x}) = F_a(\tilde{\lambda}) \leq k \;.$$
$$x \epsilon P$$

For the problem P2) this implies $\tilde{\lambda} \leq \lambda^*$.

We now turn to the important special class of convex polaroid sets, there the intersection problem plays a fundamental role in the construction of algorithms (see [21]).

Corollary 1.1: If the polaroid function $f(x,y)$ is quasi-convex in y , $\forall x \epsilon P$, then $\forall a \epsilon R^n$ the intersection function $F_a(\lambda)$ is quasi-convex in λ .

Proof: By definition one has $\forall a \epsilon R^n$:

$$F_a(\lambda) \geq f(x, y = \bar{x} - \lambda a) \quad , \quad \forall x \epsilon P$$

Let $y^1 = \bar{x} - \lambda^1 a$, $y^2 = \bar{x} - \lambda^2 a$, $\lambda^3 = \mu\lambda^1 + (1-\mu)\lambda^2$ with $\mu \epsilon [0,1]$:

$$y^3 = \bar{x} - [\mu\lambda^1 + (1-\mu)\lambda^2]a = \mu y^1 + (1-\mu)y^2 \quad ;$$

for $i=1,2,3$ one has by hypothesis $F_a(\lambda^1) \geq f(x,y^1)$, $\forall x \epsilon P$, with $F_a(\lambda^1) = f(x^1,y^1)$, for some $x^1 \epsilon P$;

hence $F_a(\lambda^3) = f(x^3,y^3) = f(x^3, \mu y^1 + (1-\mu)y^2) \leq \max \{f(x^3,y^1), f(x^3,y^2)\}$

$$\leq \max \{f(x^1,y^1), f(x^2,y^2)\} = \max \{F_a(\lambda^1), F_a(\lambda^2)\}$$

Q.E.D.

Theorem 3: Let $f(x,y)$ be quasi-convex in y , $\forall x \epsilon P$; and

$$\text{let} \quad \lambda^* = \max_{\lambda > 0} \{\lambda \mid F_a(\lambda) \leq k\}$$

then $F_a(\lambda) \leq F_a(\lambda^*) = k$, $\forall \lambda \epsilon [0, \lambda^*]$

Proof: By hypothesis $\bar{x} \epsilon P^*(k)$, implying $f(x,\bar{x}) \leq k$, $\forall x \epsilon P$ thus $F_a(0) \leq k$; the quasi-convexity of F_a established in the corollary 1.1 completes the proof.

Q.E.D.

The above theorem corroborates a well-known property of convex sets with respect to their intersection by a one-dimensional line (ray); the situation is complicated here by the fact that the function $F_a(\lambda)$ is not explicitly known but merely defined in terms of the (arbitrary) optimization problem Pl; naturally the determination of an intersection point could also be formulated as one single optimization problem in the variables x and λ ; however, the independence of the constraints in x and λ variables respectively as

well as the separate properties of f (with respect to x and y respec-
tively) motivates a separate treatment of the optimization in x and in λ ;
the result of theorem 3 indicates that the parametric search problem P2 can
be solved by increasing stepwise the value of λ until λ^* (or $\tilde{\lambda} \leq \lambda^*$)
is reached; each step consists in a (post-) optimization of the mathematical
programming problem P1. The resulting algorithm becomes a <u>non-linear descent</u>
algorithm of a particular type.

<u>Algorithm</u>:

> <u>Step 0</u>: Let $\lambda_0 = 0$, $i = 1$; choose $\lambda_1 > 0$ such that
>
> $$f(x, \bar{x} - \lambda_1 a) \leq k \quad , \quad \forall x \epsilon P , \quad \text{i.e.} \quad \bar{x} - \lambda_1 a = y_1 \epsilon P^*(k)$$
>
> <u>Step 1</u>: Solve the (linearly constrained) mathematical programming
> problem
>
> $$P1: \quad \max z(x) = f(x, \bar{x} - \lambda_i a) \quad , \quad \text{s.t.} \quad x \epsilon P$$
>
> <u>Step 2</u>:
>
> > 2.1: If $\bar{z} = \max_{x \epsilon P} z(x) = k$ then $\boxed{\text{STOP}}$
> >
> > 2.2: If $\bar{z} > k$, then choose $\lambda_{i+1} \epsilon (\lambda_{i-1}, \lambda_i)$
> >
> > 2.3: If $\bar{z} < k$, then choose $\lambda_{i+1} > \lambda_i$
> >
> > set $i = i + 1$ and go to Step 1.

<u>Remarks</u>:

1) The above algorithmic principle merely represents an unpolished set of guide-
   lines; depending on the particular problem at hand it can be refined to
   increase computational efficiency (see section 3).

2) Clearly the above algorithm need not be finite; practically however, one
   should remember that there usually is no need for an accurate value of λ^* ;

thus the exit criterion of step 2 normally will be made to contain a (often large) "fuzz" factor $\epsilon > 0$ and reads:

> 2.1: If $\bar{z} \epsilon [k-\epsilon, k]$ then STOP

and 2.3 is modified accordingly to become

> 2.3: If $\bar{z} < k-\epsilon$ then choose $\lambda_{i+1} > \lambda_i$

The algorithm then delivers an approximate intersection point $\tilde{u} \epsilon P^*(k)$ in a finite number of iterations (this number clearly depends on the quantity ϵ).

3) Some Examples of Polaroid Intersections

This section presents below a list of particular polaroids, ranked by order of increasing complexity of the corresponding intersection problems.

3.1) Bilinear polaroids: $f(x,y) = Ax + By + y^T Cx$

In this case the polaroid $P^*(k)$ is convex and the determination of an intersection point u^* is obtained by solving a parametric linear program.

$$\max z(x) = B(\bar{x}-\lambda a) + [A + (\bar{x}-\lambda a)^T C] x$$

$$\text{s.t. } x \epsilon P$$

Problem P1 is an ordinary L.P. and the increment of λ in step 2 of the algorithm is determined by the previous optimal solution $\bar{x}$ of step 1. Because $F_a(\lambda)$ is known to be piece-wise linear, the optimal value λ^* can be obtained by linear algebra in a finite number of iterations.

3.1.1.: For the case where C is a symmetric positive definite matrix, Balas has first recognized in [3] the possibility to use sets he calls outer-polars to generate

valid cutting planes in integer programming with the
help of parametric linear programming (the outer-
polar is a (generalized) polar set (see [15]) and also
a (special) polaroid set [9]). The same technique is used
by Balas and Burdet in [4] to solve quadratic concave
programs.

3.1.2: For indefinite as well as semi-definite quadratic
programs, polaroid cuts have been implemented success-
fully in a facial decomposition schema for solving the
general quadratic programming problem [21,22].

3.1.3: Another application for valid cuts in integer program-
ming is given in [7]; there polaroid cuts are constructed
from a homogeneous ($A=B=0$) bilinear polaroid function
which is centered at the origin (i.e. at $\bar{x}$); this
case presents computational advantages because the
optimization problem P1 and P2 can be combined in a
single linear program (not parametric); indeed, one
can verify by inspection that the problem P1,

$$\text{i.e., } \max_{x \epsilon P} -(\lambda a^T Cx) \leq k \text{ ,}$$

directly delivers the optimal value

$$(\lambda^*)^{-1} = \frac{1}{k} \max_{x \epsilon P} (-a^T Cx)$$

of problem P2.

3.2) The previous bilinear polaroids are a particular case of the fol-
lowing family (with parameter α):

$$f(x,y) = g(x) + \alpha(y-x)^T F(x) \text{ , where } g(x) = x^T F(x) \text{ ,}$$

(F is a vector valued function).

As an illustration, consider the (quadratic) case where

$$F(x) = Cx \quad (C = n \text{ by } n \text{ symmetric matrix)} \text{ ;}$$

one has $f(x,y) = (1-\alpha) x^T Cx + \alpha y^T Cx$

(For simplicity, let us assume C positive definite.)

3.2.1: $\alpha = 1$: $f(x,y) = y^T Cx$, bilinear polaroid (see 3.1 above)

3.2.2: $\alpha = 2$: $f(x,y) = -x^T Cx + 2y^T Cx$.

Note that $\alpha F(x) = 2 Cx = \nabla g(x)$;

Thus we have here an illustration (quadratic) of the general polaroid function $f(x,y) = g(x) + (y-x)^T \nabla g(x)$. The polaroid set $P^*(k)$ provides here for an _analytical characterization_ of the _level set of the maximal convex (concave) extension_ of the _convex (concave) function_ $g(x)$ _with respect to the set P_ , which has been characterized, in geometrical terms, by Hoang Tuy in [16].

3.2.3: $\alpha > 1$: In this case (which contains, in particular, the previous one 3.2.2), the mathematical programming problem Pl is a _convex quadratic programming problem over a polyhedral feasible set P_ and it can be solved by the corresponding classical algorithms.

3.2.4: $\alpha = \infty$: As α becomes very large, only points y with $(y-x) = 0$ will become tolerable; the polaroid set therefore simply becomes identical with the original set P .

3.2.5: $\alpha < 1$: Here problem Pl is a _concave program_ (i.e. max(min)imization of a convex (concave) objective function).

When $0 < \alpha < 1$, an illustration of the unifying insight provided by the polaroid approach in the fields of concave and/or integer programming can be found in [7].

We describe here a more general situation:

assume that $\Delta = g(\tilde{x}) = \max\limits_{x \in P} x^T C x$; then, for a small enough value $\tilde{\alpha} > 0$, the polaroid $P_{\tilde{\alpha}}^*(\Delta)$ will be the halfspace(s) determined by the hyperplane(s) supporting P at the <u>optimal solution(s)</u> $\tilde{x}$. This is the "largest" polaroid P^* in the family, with the desirable property that $\tilde{x} \notin \text{Int } P^* = (P^* - \text{bd } P^*)$, as shown in the proposition 3 below. Larger values $\alpha > \tilde{\alpha}$ yield a lower bound for the intersection parameter λ , which can be used as an approximation; the intersection algorithm could thus conceivably be refined by changing the parameter α (starting with, say, the L.P. value $\alpha = 1$) and post-optimizing the (new) problem P2 to find a better value for λ ; ultimately this process determines the intersection parameter $\bar{\lambda}$ with the polaroid $P_{\tilde{\alpha}}^*(\Delta)$.

In integer programming, the outer-domain theory (see, for instance, [6] and [7]) indicates that $P_{\tilde{\alpha}}^*(\Delta)$ is the intersection of a collection of halfspaces, each belonging to a <u>feasible vertex of the unit cube</u>; decreasing the parameter α (down from $\alpha = 1$) therefore corresponds to introducing additional <u>integrality requirements</u> into the linear program obtained in 3.1, see [7].

<u>3.2.6</u>: $\alpha = 0$: In this case $f(x,y) = g(x)$ and is independent of y ; hence P^* is the whole space whenever $k \in f(P, \mathbb{R}^n)$, and $P^* = \emptyset$ otherwise.

<u>Proposition 3</u>: Define $P_\alpha^*(k) = \{y \mid x^T Cx + \alpha(y-x)^T Cx \leq k, \quad \forall x \epsilon P\}$;

Assume $\alpha \geq 0$ and $P \subset lev_k g(x)$, that is: $g(x) = x^T Cx \leq k, \forall x \epsilon P$.

Then $P_{\alpha_1}^*(k) \subset P_{\alpha_2}^*(k)$ iff $\alpha_1 \geq \alpha_2$.

<u>Proof</u>: Take $y_1 \epsilon$ bd $P_{\alpha_1}^*(k)$ and choose $x_1 \epsilon P$ such that

$f(x_1,y_1) = x_1^T Cx_1 + \alpha_1(y_1-x_1)^T Cx_1 = k$;

then one has $(y_1-x_1)^T Cx_1 \geq 0$ because $0 \leq \alpha_1 = \dfrac{k - x_1^T Cx_1}{(y_1-x_1)^T Cx_1}$,

where $x_1^T Cx_1 \leq k$ by hypothesis.

Furthermore $y_1 \epsilon P_{\alpha_2}^*(k)$ implies $\alpha_2 \leq \alpha_1$, as shown below:

$x_1^T Cx_1 + \alpha_2(y_1-x_1)^T Cx_1 = x_1^T Cx_1 + \alpha_1(y_1-x_1)^T Cx_1 + (\alpha_2-\alpha_1)(y_1-x_1)^T Cx_1 \leq k$

hence $(\alpha_2-\alpha_1) \leq 0$ must hold true

since $x_1 \epsilon P$ and $x_1^T Cx_1 + \alpha_1(y_1-x_1)^T Cx_1 = k$, with $(y_1-x_1)^T Cx_1 \geq 0$.

On the other hand, assuming $\alpha_2 \leq \alpha_1$, suppose $y_1 \notin P_{\alpha_2}^*(k)$;

then $\exists x_2 \epsilon P$ such that

$$x_2^T Cx_2 + \alpha_2(y_1-x_2)^T Cx_2 = k' > k;$$

but $(y_1-x_2)^T Cx_2 \geq 0$ because $0 \leq \alpha_2 = \dfrac{k' - x_2^T Cx_2}{(y_1-x_2)^T Cx_2}$.

Hence one obtains from $y_1 \epsilon P_{\alpha_1}^*(k)$:

$x_2^T Cx_2 + \alpha_2(y_1-x_2)^T Cx_2 + (\alpha_1-\alpha_2)(y_1-x_2)^T Cx_2 \leq k$

that is $(\alpha_1 - \alpha_2) < 0$ must hold because

$$x_4^T Cx_2 + \alpha_2 (y_1 - x_2)^T Cx_2 > k \quad \text{by hypothesis} \quad \text{and} \quad (y_1 - x_2)^T Cx_2 \geq 0 .$$

But this implies $\alpha_1 < \alpha_2$ which is contrary to hypothesis; thus

$$y_1 \in P_{\alpha_2}^*(k) . \qquad\qquad\qquad\qquad\qquad\qquad \text{Q.E.D.}$$

One may conclude from proposition 3 that there may be a reward for solving increasingly difficult problems since for small α the intersection problem is of the same order of difficulty as the original (for instance, concave or integer) problem, one is trying to solve with the help of polaroids. Note that parametric linear programs $(\alpha = 1)$ however hold a position which seems to make them computationally quite attractive in that respect.

3.3 Simple polaroids: This class contains all the previous cases and is characterized by functions $f(x,y)$ which are <u>linear in y</u>, $\forall x \in P$; the convexity of the corresponding polaroid sets $P^*(k)$ is readily verified (see theorem 1).

Of course this class may contain a variety of different types of optimization problems P1: convex, concave, discrete, etc.. . . but linearity in y makes it easier to predict adequate increment for λ in the algorithm.

One should also note that the convexity of P^* makes a cutting plane approach possible for problems which do not possess convex level sets of the objective function $g(x)$. An example of this type can be found in [21] and a general discussion of <u>polar programs</u> is presented in [10].

3.4 A more general class yet is defined by polaroid functions $f(x,y)$ which are merely required to satisfy the hypothesis of theorem 1 in order to yield convex polaroid sets. Here, except for very special functions, there are no artifices available to render the problem P2 more easily tractable, and one has to resort to the stepwise incremental method described in the algorithm to solve the intersection problem (depending on the actual function f it may become more or less difficult to determine adequate increments $(\lambda_{i+1} - \lambda_i)$).

3.5 One may also use polaroid sets which are not assumed in advance to be convex; after solving the intersection problems involved by a cutting plane approach, for instance, one has then to test the validity of the cut; this task is essentially different from checking a quasi-convex property of $f(x,y)$: and in some instances, cuts may a posteriori be easily proved valid; when generated from non-convex polaroids, they may also be deeper than intersection cuts from convex outer-domains.

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